

On k -Convex Polygons*

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Abstract

We introduce a notion of k -convexity and explore polygons in the plane that have this property. Polygons which are k -convex can be triangulated with fast yet simple algorithms. However, recognizing them in general is a 3SUM-hard problem. We give a characterization of 2-convex polygons, a particularly interesting class, and show how to recognize them in $O(n \log n)$ time. A description of their shape is given as well, which leads to Erdős-Szekeres type results regarding subconfigurations of their vertex sets. Finally, we introduce the concept of generalized geometric permutations, and show that their number can be exponential in the number of 2-convex objects considered.

1 Introduction

The notion of convexity is central in geometry. As such, it has been generalized in many ways and for different reasons. In this paper we consider a simple and intuitive generalization of convexity, which to the best of our knowledge has not been worked on. It leads to an appealing class of polygons in the plane with interesting structural and algorithmic properties.

A set in \mathbb{R}^d is convex if its intersection with every straight line is connected. This definition may be relaxed to directional convexity or D -convexity [15, 23], by considering only lines parallel to one out of a (possibly infinite) set D of vectors. A special case is

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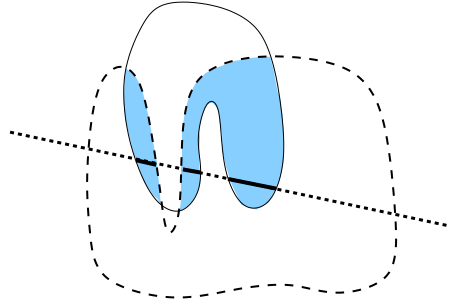


Figure 1: Intersecting two 2-convex sets

ortho-convexity [27], where only horizontal and vertical lines are allowed. For any fixed D , the family of D -convex sets is closed under intersection, and thus can be treated in a systematic way using the notion of *semi-convex* spaces [29], which is sometimes appropriate for investigating visibility issues. The D -convex hull of a set M is the intersection of all D -convex sets that contain M . If D is a finite set, this definition of a convex hull may lead to an undesirably sparse structure—an effect which can be remedied by using a stronger, functional (rather than set-theoretic) concept of D -convexity [23].

k -Convex Sets We consider a different generalization of convexity. A set M in \mathbb{R}^d is called *k -convex (with respect to transversal lines)* if there exists no straight line that intersects M in more than k connected components. Throughout the paper we will use the term *k -convex*, for short¹. Note that 1-convexity refers to convexity in its standard meaning. To reformulate in terms of visibility, call two points $x, y \in M$ *k -visible* if $\overline{xy} \cap M$ consists of at most k components. Thus a set is k -convex if and only if any two of its points are mutually k -visible. Applications of this concept may arise from placement problems for modems that have the capacity of transmitting through a fixed number of walls [1, 13].

Unlike directional convexity, k -convexity fails to show the intersection property: The intersection of k -convex sets is not k -convex in general (for fixed k). Figure 1 gives an example. For $k \geq 2$, a k -convex set M may be disconnected, or if connected, its boundary may be disconnected. In this paper, we will restrict attention (with an exception in Section 4) to simply connected sets in two dimensions, namely, simple polygons in the plane.

There are two notions of planar convexity that appear to be close to ours. One is *k -point convexity* [6, 32] which requires that for any k points in a set M in \mathbb{R}^2 , at least one of the line segments they span is contained in M . Thus 2-point convex sets are precisely the convex sets. The other is *k -link convexity* [22], being fulfilled for a given polygon P if, for any two points in P , the geodesic path connecting them inside P consists of at most k edges. The 1-link convex polygons are just the convex polygons. While there is a

¹We face notational ambiguity. The term ‘ k -convex’ has, maybe not surprisingly, been used in different settings, namely, for functions [26], for graphs [5], and for discrete point sets [20, 33]. Also, the concept of k -point convexity [32] has later been called k -convexity in [6].

relation between k -convexity and the former concept (as we will show in Section 2), the latter concept is totally unrelated.

We will study basic properties of k -convex polygons, in comparison to existing polygon classes and convexity concepts in Section 2. This offers an alternative to the approach in [4] to define ‘realistic’ polygons as those being guardable (visible) by at most k guards. We prove that given a simple polygon P , the problem of finding the smallest k such that P is k -convex (equivalently, to find the stabbing number of P) is 3SUM-hard. On the other hand, a recognition algorithm that runs in $O(n^2)$ time for a polygon with n vertices is easy to obtain. Interestingly, k -convex polygons can be triangulated, by a quite simple method, in $O(n \log k)$ time. An $O(nk)$ time complexity is achieved in [4] for k -guardable polygons.

The first nontrivial value, $k = 2$, deserves particular attention. Already in this case, a novel class of polygons is obtained. A characterization of 2-convex polygons is given in Section 3. It leads to an $O(n \log n)$ time algorithm for recognizing such polygons. Note that 2-convex polygons add to the list of special classes of polygons [4, 11] that allow for simple $O(n)$ time triangulation methods. We also provide a qualitative description of their shape, which implies an Erdős-Szekeres type result, namely, that every 2-convex polygon with n vertices contains a subset of at least \sqrt{n} vertices in convex position, and that its vertex set can be decomposed into at most $2\sqrt{2n}$ subsets in convex position.

In Section 4, we turn our attention to general k -convex domains. We give observations on the union and intersection properties of such domains, and elaborate on an attempt to generalize the notion of geometric permutations from convex sets to k -convex sets. In contrast to the $O(m)$ bound in [10] on the number of geometric permutations of m convex sets, it turns out that the number of generalized geometric permutations can be exponential in m , already for 2-convex sets. For 2-convex *polygonal* domains, the number of generalized geometric permutations is $O(n^2)$, if n denotes the total number of their vertices.

Various open questions are raised by the proposed concept of k -convexity. We list those which seem most interesting to us, along with a brief discussion of our results, in Section 5.

2 k -Convex Polygons

2.1 Basic properties

We start with exploring some basic properties of k -convex polygons, and compare them to existing polygon classes and related concepts. All geometric objects we will consider are closed sets in the Euclidean plane.

Let P be a simple polygon, and denote by n the number of vertices of P . Two line segments e and e' are said to *cross* if $e \cap e'$ is a point in the relative interior of both e and e' . Clearly, a polygon P is k -convex if every line segment with endpoints in P crosses at most $2(k - 1)$ edges of P . The *stabbing number* [14, 35] of a set of (interior-disjoint) line segments is the largest possible number of intersections between the set and a straight line. A polygon is k -convex if and only if its stabbing number is at most $2k$. Thus, all

our observations on k -convexity could be reformulated in terms of stabbing numbers. Moreover, we have the following property, which we state for future reference.

Lemma 1. *If P is a k -convex polygon, then any polygon living on a subset of vertices of P is also k -convex.*

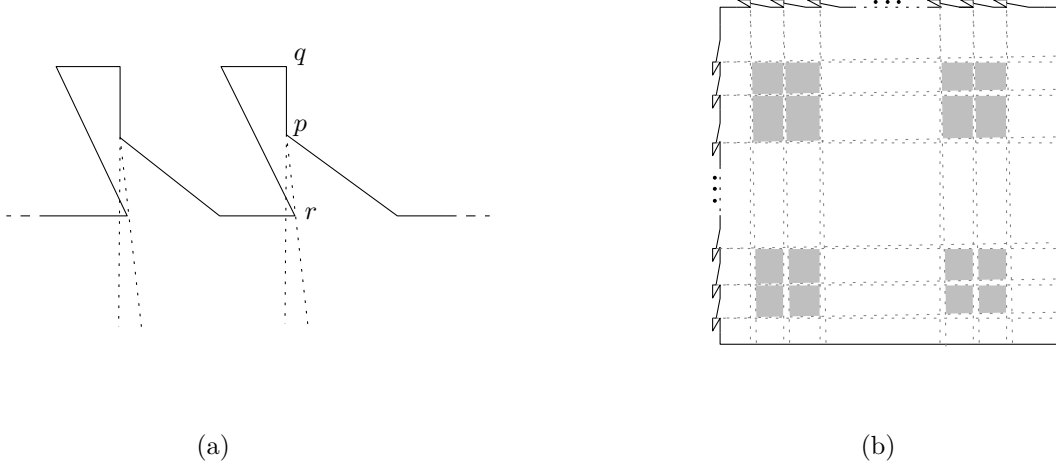


Figure 2: Quadratic 2-kernel construction

The kernel of a simple polygon P is the set of points that see all the polygon. Its generalization to k -convexity shows that 2-convexity is already significantly more complex than standard convexity. The k -kernel of P , denoted as $M_k(P)$, is the set of points from which the entire polygon P is k -visible. Note that P is k -convex if and only if $P = M_k$. While M_1 is known to be a convex set which is computable in $O(n)$ time [21], M_2 may have $\Omega(n^2)$ complexity: If we consider the ‘spike’ in Figure 2(a), the wedge between the lines pq and pr is not part of M_2 . If we arrange such spikes along the boundary of a rectangle, as in Figure 2(b), we get a quadratic number of disconnected areas which are part of the 2-kernel. Therefore, any algorithm computing M_2 , or trying to check 2-convexity via the comparison of P and M_2 would have $\Omega(n^2)$ complexity in the worst case.

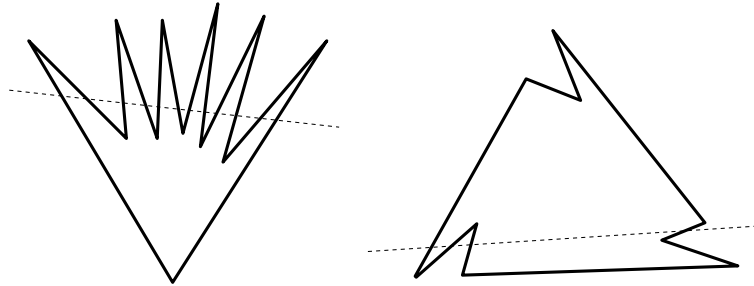


Figure 3: Star-shaped versus 2-convex

There is also no immediate relation to *star-shaped* polygons, i.e., polygons P with $M_1 \neq \emptyset$. Figure 3 shows a polygon on the left hand side which is star-shaped but only $\frac{n}{2}$ -convex.

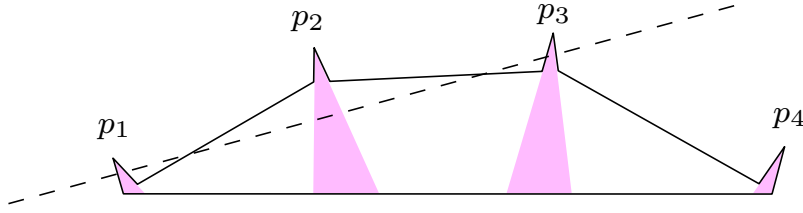


Figure 4: Guarding a 3-convex polygon

On the right hand side, we see a polygon which is 2-convex but not star-shaped. Visually, 2-convexity seems to be closer to convexity than is star-shapedness. Note that cutting a 2-convex polygon with any straight line leaves (at most) three parts, each being 2-convex itself. This is not true in general for star-shaped polygons.

While 2-convexity obviously restricts the winding number of a polygon [34], its link distance [22] is unaffected and may well be $\Theta(n)$. Conversely, a polygon which is 2-link convex (such that any two of its points are at link distance 2 or less) may fail to be k -convex for sublinear k . The star-shaped polygon in Figure 3 (left) is an example.

There is, interestingly, a relation to k -point convexity as defined in [32]. A polygon P is called k -point convex if for any k points p_1, \dots, p_k in P , at least one of the closed segments $p_i p_j$ belongs to P . Every k -point convex polygon P is $(k - 1)$ -convex. To verify this, we prove that if P is not $(k - 1)$ -convex, then P is not k -point convex: Because P is not $(k - 1)$ -convex, there exists a line ℓ which intersects P in at least k components. If we select a point in each component, it is clear that none of the segments defined by them is inside P , and therefore P is not k -point convex. However, no implication exists in the other direction. For example, the 2-convex polygon in Figure 5 fails to be k -point convex for $k < \frac{n}{3}$. Also, any k -point convex polygon can be expressed as the union of m convex polygons, where m depends (exponentially) on k but is independent of the polygon size n ; see [6]. Such a property is not shared by k -convex polygons, as can be seen from the 2-convex polygon in Figure 5.

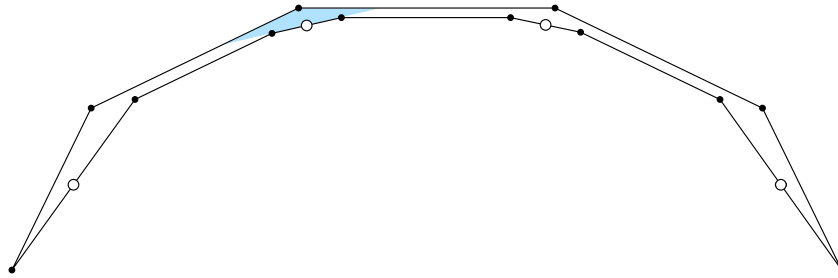


Figure 5: Every white dot requires a different guard

The class of k -convex polygons also differs from the class of k -guardable polygons defined in [4]. It is known that any simple polygon with n vertices can be guarded with at most $\lfloor \frac{n}{3} \rfloor$ guards [25, 31]. The example in Figure 5 shows that this number of guards can be already necessary for 2-convex polygons. This is one of the reasons why most

tools developed to study guarding problems of polygons are not very useful in the study of modern illumination problems [1].

Pseudo-triangles are polygons with exactly three convex vertices, joined by three reflex side chains. Any pseudo-triangle is 2-convex: If a straight line crosses a side chain twice, then it can cross each of the remaining two side chains at most once (Figure 6, left). That is, the stabbing number of a pseudo-triangle is four or less. In the same way as a triangulation defines a partition of the underlying domain into convex polygons, any pseudo-triangulation [28] or any pseudo-convex decomposition [3] gives a partition into 2-convex polygons. It is an open problem (see Problem(e) in the final section) whether it is possible to subdivide a polygon with n vertices into a sublinear number of 2-convex polygons. If Steiner points are disallowed, then a 2-convex partition may have to consist of $\Theta(n)$ parts; see Figure 6 (right).

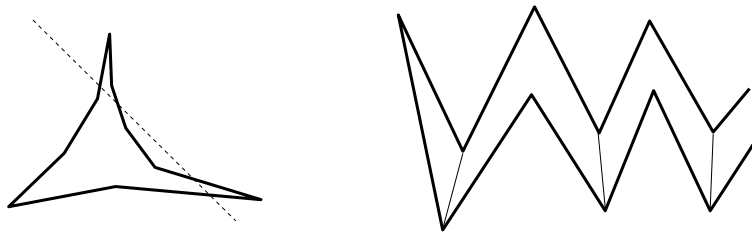


Figure 6: Pseudo-triangle and 2-convex partition

Two other natural questions for k -convex n -gons are decomposing them into few convex pieces in terms of k , as well as giving a bound on the number of pieces of the convex hull minus the polygon. However, in both cases the answer is unrelated to k , because the polygon in Figure 7 (left) is 2-convex yet requires $n - 2$ convex pieces, which is obviously tight because every polygon can be triangulated. On the other hand, the polygon in Figure 7 (right) is also 2-convex and has $\frac{n}{2}$ pockets, which is tight because two consecutive points on the convex hull are at least two edges apart if they are not adjacent on the polygon boundary.

2.2 Recognition complexity

Let us start our considerations by observing that the stabbing number of a polygon with n vertices can be easily found in $O(n^2)$ time, as follows. The standard duality transform maps each edge of the polygon to a double wedge consisting of lines through a common point, not including the vertical; the two lines that bound this dual wedge correspond to the endpoints of the primal segment. In the primal, any point inside a wedge is a line that stabs the segment. Therefore, the primal line that would stab most segments is in the dual a point that belongs to as many double wedges as possible—a maximum depth problem that can easily be solved by constructing the arrangement and then traversing its cells.

The obtained $O(n^2)$ time bound is essentially tight, as in this section we prove that finding the stabbing number of a polygon, or, equivalently, finding the smallest k for which the polygon is k -convex, is a 3SUM-hard problem. This family of problems is

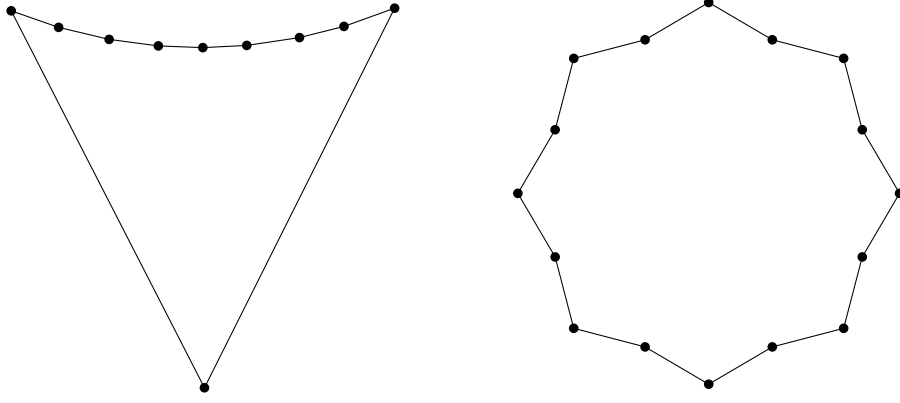


Figure 7: A polygon with many convex pieces (left) and many pockets (right).

widely believed to have an $\Omega(n^2)$ lower bound for the worst case runtime [17, 19]. We start by giving the following result, which follows directly from Theorem 4.1 in [17].

Lemma 2. *For every integer a , let us consider the point $p_a = (a, a^3)$ on the cubic $y = x^3$. Then if a, b, c are distinct integers, p_a, p_b, p_c are collinear if and only if $a + b + c = 0$.*

Proof. The points p_a, p_b, p_c are collinear if and only if the determinant

$$\begin{vmatrix} a & a^3 & 1 \\ b & b^3 & 1 \\ c & c^3 & 1 \end{vmatrix} = (b - a)(c - a)(c - b)(a + b + c)$$

vanishes, which, the numbers being different, happens exactly when $a + b + c = 0$. \square

We next show that the points p_x with $x \in \mathbb{Z}$ on the cubic $y = x^3$ can be replaced by infinitesimally small vertical segments S_x with upper endpoint p_x , such that three of them can be stabbed by a single line if and only if their three upper endpoints are collinear.

Lemma 3. *Let m, a, b, c, M be five integers such that $m < a < b < c < M$. Let $\varepsilon = \frac{1}{6(M-m)}$ and let s_t be the (vertical) segment with endpoints $p_t = (t, t^3)$ and $p'_t = (t, t^3 - \varepsilon)$. Then s_a, s_b and s_c can be stabbed by a single line if and only if p_a, p_b , and p_c are collinear.*

Proof. Assume that the points p_a, p_b, p_c are not collinear, and let us take three points $q_a = (a, a^3 - \varepsilon_1), q_b = (b, b^3 - \varepsilon_2), q_c = (c, c^3 - \varepsilon_3)$, with $0 \leq \varepsilon_1, \varepsilon_2, \varepsilon_3 \leq \varepsilon$, i.e., three points on the segments s_a, s_b and s_c . The points q_a, q_b, q_c would be collinear if and only if the determinant

$$\begin{vmatrix} a & a^3 - \varepsilon_1 & 1 \\ b & b^3 - \varepsilon_2 & 1 \\ c & c^3 - \varepsilon_3 & 1 \end{vmatrix} = \begin{vmatrix} a & a^3 & 1 \\ b & b^3 & 1 \\ c & c^3 & 1 \end{vmatrix} + \begin{vmatrix} a & -\varepsilon_1 & 1 \\ b & -\varepsilon_2 & 1 \\ c & -\varepsilon_3 & 1 \end{vmatrix} =$$

$$= \underbrace{(b - a)(c - a)(c - b)(a + b + c)}_z + \underbrace{\varepsilon_1(b - c) - \varepsilon_2(a - c) + \varepsilon_3(a - b)}_\delta$$

is 0, but this is impossible because by Lemma 2, z is an integer different from 0, which cannot become 0 by the addition of δ , because

$$|\varepsilon_1(b-c) - \varepsilon_2(a-c) + \varepsilon_1(a-b)| \leq |\varepsilon_1(b-c)| + |\varepsilon_2(a-c)| + |\varepsilon_1(a-b)| \leq \varepsilon \cdot 3(M-m) = \frac{1}{2}.$$

□

Theorem 1. *The problem of finding the stabbing number of a polygon is 3SUM-hard.*

Proof. The 3SUM problem is defined as follows: Given a set S of n integers, do there exist three elements $a, b, c \in S$ such that $a + b + c = 0$? We will prove below that this problem can be reduced in $O(n \log n)$ time to the problem of computing the stabbing number of an n -gon. In other words, using the notation in [17],

$$3\text{SUM} \lll_{O(n \log n)} \text{stabbing number of a polygon}.$$

Let x_1, \dots, x_n be the input integers. We proceed to the reduction by steps.

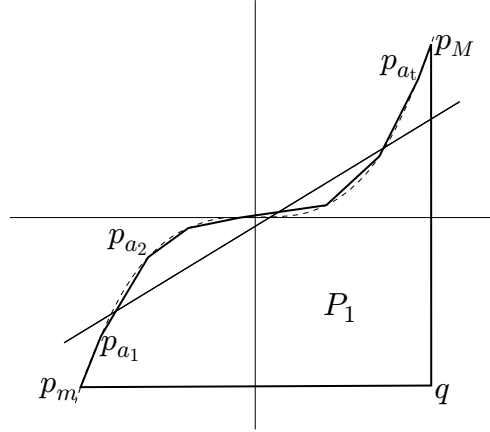


Figure 8: Polygon P_1 with vertices on the cubic $y = x^3$ (dashed). The scale of the axis is not 1:1, to make the figure visible.

Step 1. Sort the input numbers; let $y_1 \leq y_2 \leq \dots \leq y_n$ be the resulting list, \mathcal{L} . This step is done in $O(n \log n)$ time.

Step 2. If 0 appears thrice in \mathcal{L} , exit with a sum of three numbers in the input being 0, otherwise continue. The step is completed in linear time.

Step 3. If $a \neq 0$ appears at least twice in \mathcal{L} , check whether $-2a \in \mathcal{L}$. If so, exit with a sum of three numbers in the input being 0, otherwise continue. The step is completed in $O(n \log n)$ time, as binary search can be used in the sorted list (in fact, $O(n)$ time is sufficient by scanning the list from left to right and from right to left, in a coordinated simultaneous advance).

Step 4. Remove multiples from the list \mathcal{L} so that each number appears exactly once. This requires linear time. Let $a_1 < a_2 < \dots < a_t$ (where $t \leq n$) be these numbers.

Step 5. Define $m = a_1 - 1$, $M = a_t + 1$, and $q = (M, m^3)$. Now let us consider the polygon P_1 whose vertices, described clockwise, are $p_m p_{a_1} p_{a_2} \dots p_{a_t} p_M q$, where $p_x = (x, x^3)$, as in

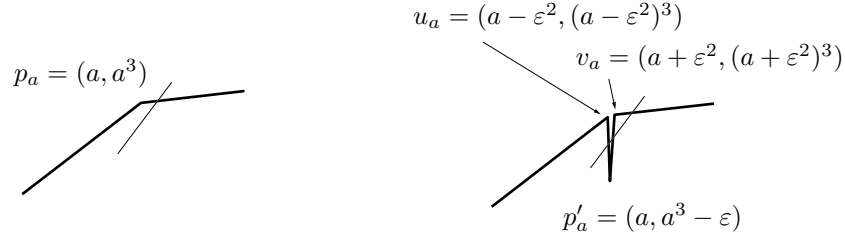


Figure 9: Polygon P_2 with vertices on the cubic $y = x^3$ replaced by infinitesimal vertical slots.

the preceding lemma (Figure 8). Observe that the stabbing number of P_1 is 4, and that the polygon can be constructed in $O(n)$ time.

Step 6. Next we modify the polygon P_1 to become the polygon P_2 whose vertices, described clockwise, are $p_m u_{a_1} p'_{a_1} v_{a_1} u_{a_2} p'_{a_2} v_{a_2} \dots u_{a_t} p'_{a_t} v_{a_t} p_M q$, where $p'_x = (x, x^3 - \varepsilon)$, $u_x = (x - \varepsilon^2, (x - \varepsilon^2)^3)$, $v_x = (x + \varepsilon^2, (x + \varepsilon^2)^3)$; see Figure 9. This polygon can be constructed in $O(n)$ time, and in the vicinity of the point p_{a_i} its stabbing number changes locally from 1 to 3; see Figure 10. Let s_{a_i} be the segment joining u_{a_i} to p'_{a_i} . Therefore, the stabbing number of P_2 is 12 if and only if three of the segments s_{a_i} can be simultaneously stabbed, and 8 otherwise, as two of those segments can always be stabbed.

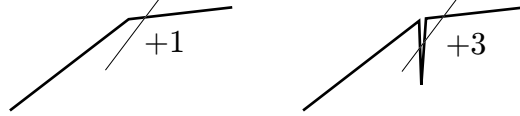


Figure 10: From P_1 to P_2 the stabbing number locally increases from 1 to 3.

Step 7. Compute the stabbing number of polygon P_2 using the best possible algorithm.

Step 8. If the stabbing number of polygon P_2 is 10, conclude that there were three numbers in the initial input such that their sum is 0; if the stabbing number of P_2 is 8, conclude that there are no such three numbers.

The correctness of Step 8 is a direct consequence of Lemma 2 and Lemma 3. As all steps but Step 7 have overall complexity $O(n \log n)$, we conclude that Step 7, the computation of the stabbing number of a polygon, is a 3SUM-hard problem, as claimed. \square

As an immediate consequence we obtain:

Corollary 1. *The problem of deciding the smallest number k such that a given polygon is k -convex is 3SUM-hard.*

Also, if we replace Step 7 in the proof of Theorem 1 by checking whether or not polygon P_2 is 4-convex, we get:

Corollary 2. *The problem of deciding whether a given polygon is 4-convex is 3SUM-hard.*

2.3 Fast triangulation

Triangulating a simple polygon faster than in $O(n \log n)$ time with a simple method is a challenging open problem. For k -convex polygons, this can be achieved, by the fact that we can sort the vertices of a k -convex polygon P in any given direction (say, x -direction) in $O(kn)$ time: Simply scan around ∂P and use insertion sort, starting each time from the place where the x -value of the previous vertex has been inserted. Then any fixed value x_j , once being inserted, takes part in later comparisons at most $2k - 1$ times because otherwise, the vertical line $x = x_j$ would intersect P in more than k components. Having x -sorted P 's vertices, a simplified plane sweep method can be used to build a vertical trapezoidation [9, 16] (and then a triangulation) of P . Only trivial data structures may be used, as the scenario on the sweep line is of complexity $O(k)$, by the k -convexity of P . Thus, each vertex of P can be processed in $O(k)$ time during the sweep. We conclude:

Proposition 1. *Any k -convex polygon can be triangulated in $O(kn)$ time and $O(n)$ space.*

Using suitable data structures, a faster yet still implementable algorithm is possible, as we show next. Call a polygon k -monotone if every vertical line intersects the interior of the polygon in k intervals. (This property is implied by k -convexity, and is equivalent to x -monotonicity for $k = 1$.) Actually, we do not even need the polygon to be simple, we just need a sequence of x -coordinates such that every other x -coordinate comes between at most $2k$ consecutive pairs of x -coordinates.

Lemma 4. *The vertices of any k -monotone polygon can be x -sorted in $O(n \log(2 + k))$ time.*

Proof. We use a binary insertion sort, in which we add the points in order along the polygon into a balanced binary search tree. The binary search tree has the *dynamic finger property*: inserting an element that has rank r different from the previously inserted element costs only $O(\log(2 + r))$ time. (For example, splay trees [30] and Brown & Tarjan finger trees [7] both have this property.) Once the elements are inserted into the binary search tree, we simply perform a linear-time in-order traversal to extract them in sorted order.

Now we find the bound on the total cost of the insertions. When we insert an element of rank difference r from the previously inserted element, we can charge this cost to r points formed from projecting all vertices onto all edges of the polygon. There are at most $O(nk)$ such points of projection, so the total number of charges is at most $O(nk)$. Thus the total insertion cost is $O(\sum_{i=1}^n \log(2 + r_i))$ where $\sum_{i=1}^n r_i = O(nk)$. Such a sum is maximized when the r_i are all roughly equal, which means that they are all $\Theta((nk)/n) = \Theta(k)$. Therefore the total cost is at most $O(n \log(2 + k))$. \square

To see that this bound is optimal in the comparison model, consider the case in which the polygon is a comb with k tines. Then sorting the x -coordinates is equivalent to merging k sorted sequences of length n/k , which is known to take $\Theta(n \log k)$ time in the worst case.

Lemma 4 yields a fast triangulation method for general k -convex polygons. We first sort the vertices of the k -convex polygon P in a fixed direction, in $O(n \log k)$ time. Again, a plane sweep is used to compute a triangulation of P . As the intersection of P with the

sweep line is of complexity $O(k)$ only, by the k -convexity of P , each of the n vertices of P can be processed in $O(\log k)$ time during the sweep.

Theorem 2. *Any k -convex polygon can be triangulated in $O(n \log k)$ time and $O(n)$ space.*

3 Two-Convex Polygons

3.1 Characterization

In this section we give a characterization of 2-convex polygons that allows their recognition in time $O(n \log n)$, and a description of their structure that will be used later in several of our results.

We observe that k -convexity is a property that may be lost by small perturbations on the positions of the vertices of a polygon. For example, small changes in the positions of the vertices p_1, \dots, p_4 in Figure 4, could yield a 1- or 2-convex polygon. As a consequence, stabbing a polygon along its edges will not, in most cases, give enough information for deciding its k -convexity.

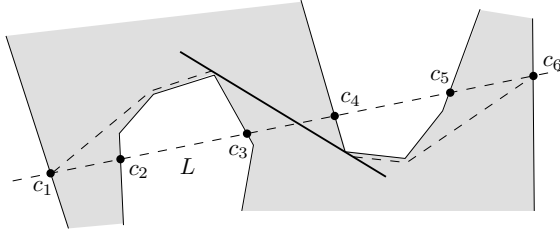


Figure 11: Constructing an inner tangent

Let P be a simple polygon, and denote its boundary by ∂P . A line L is called a j -stabber of P if L crosses ∂P at least j times. Note that a j -stabber may contain whole edges of P ; these are *not* considered to contribute to the count. An edge of a polygon P is called an *inflection edge* if it joins a convex and a reflex vertex of P . An *inner tangent* of P is a line segment $T \subset P$ that contains two non-adjacent reflex vertices of P in its relative interior; see Figure 11.

Lemma 5. *A simple polygon P is 2-convex if and only if P has no inner tangent, and no 3-stabber that contains an inflection edge.*

Proof. Consider the ‘only if’ implication first. Suppose that for some inflection edge e of P , its supporting line L is a 3-stabber for P . Then a slight perturbation of L gives rise to a line L' that crosses ∂P at least two more times than L does. One of these extra crossings lies in the interior of e , and one on the second edge of P incident to the reflex vertex of e . Thus P admits a 6-stabber, and therefore P cannot be 2-convex. Similarly, if P has some inner tangent T , defined by reflex vertices u and v , say, then a 6-stabber cutting off u and v exists in the vicinity of T .

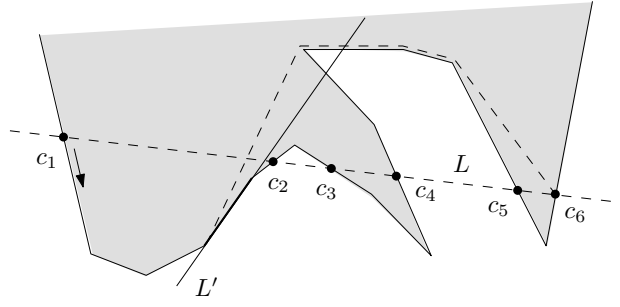


Figure 12: Constructing a 3-stabber

To prove the ‘if’ implication, assume that P is not 2-convex. Then there exists a 6-stabber L of P . Let c_1, \dots, c_6 denote six consecutive crossings of L with ∂P such that the line segment joining c_1 to c_2 is contained in P . Two types of crossing pattern arise, as shown in Figures 11 and 12, respectively. In the former case, we simply connect the two crossings c_1 and c_6 with a geodesic path inside P (dashed), and obtain an inner tangent (bold). In the latter case, if an inner tangent exists locally, then we can construct it in a similar way, possibly by moving the geodesic’s endpoints from c_1 and c_6 closer to each other along ∂P . Otherwise (as Figure 12 illustrates) we can find an inflection edge (bold) between c_1 and c_3 whose supporting line L' crosses P ’s boundary twice between c_4 and c_5 . But the line L' has to cross ∂P once more, and thus it is a 3-stabber containing an inflection edge, which is not possible. \square

3.2 Recognition

Suppose that we want to decide if a polygon P is 2-convex. (Assume that P is not convex; the problem is trivial, otherwise.) Our recognition algorithm is based on Lemma 5. We look for inner tangents and 3-stabbers at inflection edges. To this end, we first shoot two rays at each reflex vertex v of P in the directions determined by the edges of P incident with v . This can be done in $O(n \log n)$ [8]. If ∂P is intersected more than once by any of these rays, then a 6-stabber exists, and we report that P is not 2-convex.

Suppose then that each ray shot at the reflex vertices of P yields a *unique* intersection point with ∂P . We store the points of intersection and use them to check for inner tangents. Define, for each reflex vertex v of P , its *critical range* $C(v)$ as the set of all points $x \in \partial P$ such that \overline{vx} can be prolonged to a line segment tangent to P at v . Note that such a segment need not lie entirely in P . However, $C(v)$ consists of exactly two connected intervals on ∂P whose endpoints are among the stored points obtained from ray shooting. See Figure 13, where $C(v)$ is drawn with bold lines.

Lemma 6. *P admits an inner tangent if and only if P has two reflex vertices v and v' such that $v \in C(v')$ and $v' \in C(v)$.*

Proof. Clearly, if two vertices v and v' define an inner tangent for P , then $v \in C(v')$ and $v' \in C(v)$ hold. Conversely, assume that both inclusion conditions are fulfilled. Consider the geodesic path π between v and v' inside P . Either π is the line segment $\overline{vv'}$, and thus can be extended to an inner tangent at v and v' , or π detours via reflex vertices w_1, \dots, w_m .

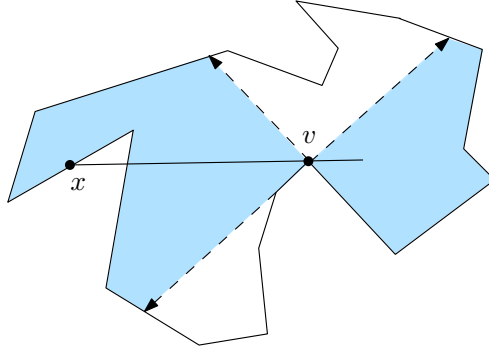


Figure 13: Critical range for vertex v

In the latter case (which is illustrated in Figure 14), we can extend the line segments $\overline{vw_1}$ and $\overline{w_mv'}$ to inner tangents, because ray shooting at reflex vertices of P has led to unique intersection points with ∂P , and thus the path π does not cross the four (dashed) rays depicted in the figure. \square

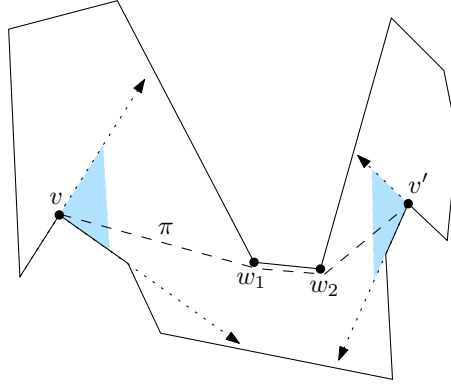


Figure 14: Geodesic path between v and v'

The strategy for detecting inner tangents is now clear. First we associate to each reflex vertex v two intervals, each containing a set of consecutive vertices of P lying within the critical range of v . We then choose a point x in ∂P and calculate the set $R(x)$ of reflex vertices containing x in their critical ranges. We then slide x along ∂P , maintaining the set $R(x)$ in a range search tree. $R(x)$ has to be updated each time x passes over a point of intersection of ∂P with a ray shot from a reflex vertex of P . Moreover, when x reaches some reflex vertex v of P , we check in logarithmic time whether the intervals previously associated to v contain any element in $R(v)$. Thus this phase takes $O(n \log n)$ time.

Since range search trees can be implemented in linear space, we have the following result:

Theorem 3. *Deciding if a simple polygon P with n vertices is 2-convex can be done in $O(n \log n)$ time and $O(n)$ space.*

3.3 Shape structure

We have given a geometric characterization of 2-convex polygons, in Subsection 3.1. The present subsection aims at giving a qualitative description of their shape.

Lemma 7. *Let P be a 2-convex polygon. Let $C = p_0p_1 \dots p_t$ be the chain of vertices that connects (counterclockwise) two consecutive vertices p_0, p_t on the convex hull $CH(P)$. Then C can be partitioned into three chains $C_1 = p_0p_1 \dots p_r$, $C_2 = p_{r+1} \dots p_s$, and $C_3 = p_{s+1} \dots p_t$, for $0 \leq r \leq s < t$, such that all vertices in C_1 and C_3 are convex (in P), while all vertices in C_2 are reflex.*

Proof. If C_2 is empty, the lemma is obviously true, so we assume that the chain C contains at least one reflex vertex. Let p_k be the last point of the chain C to be hit in a parallel sweep starting with the line defined by p_0p_t which, without loss of generality, we assume to be horizontal. Observe that p_k is necessarily a reflex vertex. Let us recall that an inflection edge is adjacent to a reflex vertex and to a convex vertex and that, according to Lemma 5, 2-convex polygons do not have 3-stabbers containing an inflection edge. We are going to see that there is at most one inflection edge in the chain $p_0 \dots p_k$, and that the same applies to the chain $p_k \dots p_t$. Let $p_i p_{i+1}$ be the first inflection edge starting from p_0 (see Figure 15.a). As the directed ray $p_i p_{i+1}$ intersects the chain $p_t p_0$ at least once, the chain $p_{i+1} p_t$ is necessarily to the right of the oriented line $p_i p_{i+1}$. Observe that p_t is to the right of the oriented line $p_i p_{i+1}$, because otherwise a rotation around p_i starting from $p_{i-1} p_i$ will define an inner tangent, forbidden in 2-convex polygons according to Lemma 5. Finally, if we assume that there exists a convex vertex p_j in the chain $p_{i+1} p_k$, then we have that $p_{j-1} p_j$ is an inflection edge supporting a 3-stabber: the ray $p_j p_{j-1}$ intersects the polygon at least once, while the ray $p_{j-1} p_j$ intersects the chain $p_k p_t$ and therefore intersects the polygon at least twice. \square

Observe that the chains C_1 and C_3 might be singletons in some cases, and that C_2 might be empty. However, the generic aspect of a pocket and the shape of a 2-convex polygon are as shown in Figure 15.b). The next result follows directly:

Corollary 3. *If the convex hull of a 2-convex polygon P has k vertices, then the boundary of P can be decomposed into $2k$ convex chains.*

The Erdős-Szekeres Theorem says that every set of n points in general position contains at least $\log n$ points that are in convex position, and that this value is asymptotically tight [12]. As every point set can be ‘polygonized’, one cannot expect a better value when the points are chosen from the set of vertices of an arbitrary polygon. However, when a point set is the set of vertices of a 2-convex polygon, we can improve this bound as follows.

Theorem 4. *Every 2-convex polygon with n vertices has a subset of $\lceil \sqrt{n/2} \rceil$ vertices in convex position. This bound is tight.*

Proof. By Corollary 3, the boundary of a 2-convex polygon with k vertices on its convex hull can be decomposed into at most $2k$ convex chains. If $k \geq \lceil \sqrt{n/2} \rceil$, we are done, otherwise one of the $2k$ convex chains necessarily has size at least $\lceil \sqrt{n/2} \rceil$. The amoeba-like example in Figure 15.b), with $k = \lceil \sqrt{n/2} \rceil$ vertices in the convex hull and $2k$ convex chains of equal size shows that this bound is tight. \square

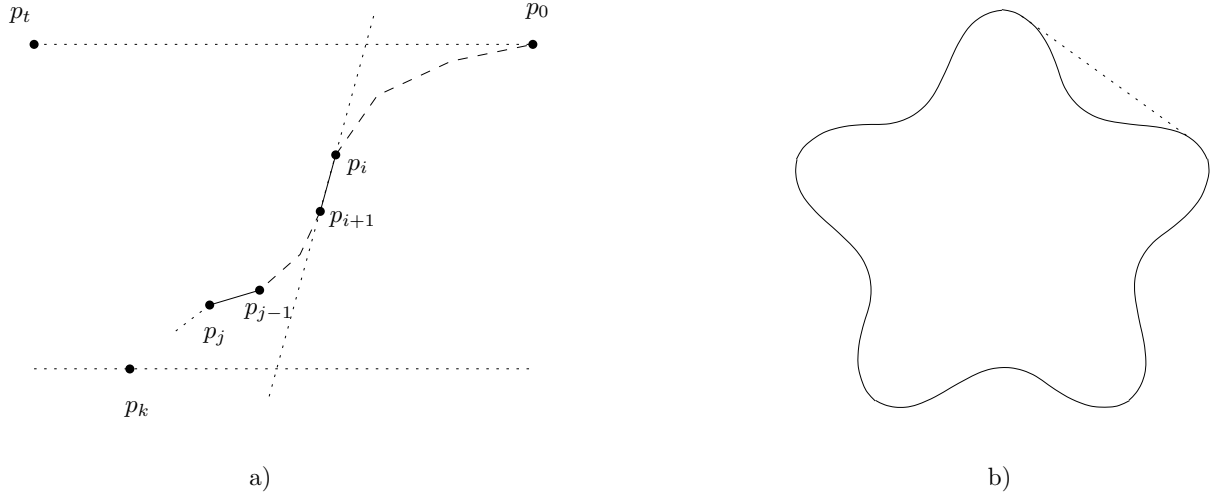


Figure 15: Illustration for the proof of Lemma 7.

We conclude this section with a consequence of the preceding theorem.

Corollary 4. *If an n -gon is 2-convex, then its vertices can be grouped into at most $2\sqrt{2n}$ subsets, each in convex position.*

Proof. Let $S(n)$ be the number of convex subsets needed to partition the vertex set of a 2-convex polygon with n vertices. We show that $S(n) \leq \alpha\sqrt{n}$ by induction over n . The induction base for $n = 3$ is obvious, and valid for any $\alpha \geq 1$. By Theorem 4 we find one convex subset of size at least $\lceil \sqrt{n/2} \rceil$. Moreover, by Lemma 1, the remaining points define also a 2-convex polygons and we have $S(n) \leq 1 + S(\lfloor n - \sqrt{n/2} \rfloor) \leq 1 + \alpha\sqrt{\lfloor n - \sqrt{n/2} \rfloor}$, where the last inequality comes from the induction hypothesis. To prove the lemma it is sufficient to show that $1 + \alpha\sqrt{n - \sqrt{n/2}} \leq \alpha\sqrt{n}$. Standard manipulation shows that this is true for any $\alpha \geq 2\sqrt{2}$ and any $n \geq 1$. \square

4 General k -Convex Domains

The union or intersection of simple polygons may not be a polygon. In view of this fact, the issue of how the degree of convexity behaves with respect to these operations is not meaningful for this class of objects. In this section, we consider larger classes of sets in \mathbb{R}^2 for which these natural questions may be discussed. We first study some properties of general (not necessary polygonal) subsets of \mathbb{R}^2 .

Lemma 8. *The union of a k -convex set Q_1 and an m -convex set Q_2 is a $(k + m)$ -convex set, which is tight.*

Proof. The number of intersections of a line with the boundary of $Q_1 \cup Q_2$ can be at most $2k + 2m$. On the other hand, if Q_1 and Q_2 are disjoint and the line that gives k and m connected components, respectively, is the same, the value is achieved. \square

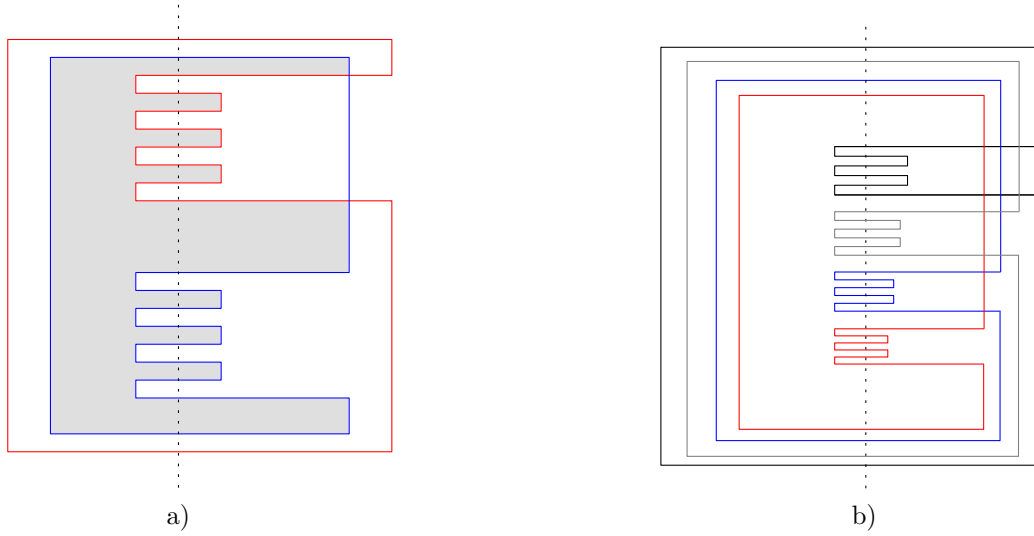


Figure 16: Intersecting k -convex sets

Lemma 9. *The intersection of a k -convex set Q_1 and an m -convex set Q_2 is a $(k+m-1)$ -convex set, which is tight.*

Proof. An oriented line will cut the boundary of Q_1 at most $2k$ times and the boundary of Q_2 at most $2m$ times. However, the first intersection point a does not contribute to the total number of cuts with $Q_1 \cap Q_2$ unless $a \in Q_1 \cap Q_2$, in which case it contributes only once instead of twice as intersection. The same happens with the last point, which gives the upper bound. An example proving the tightness appears in Figure 16(a). \square

Corollary 5. *The intersection of a family of m k -convex sets is a $(m(k-1)+1)$ -convex set, which is tight.*

Proof. The upper bound follows from the preceding theorem, and a construction giving its tightness is shown in Figure 16(b). \square

Theorem 5. *There is no Helly-type theorem for k -convex sets.*

Proof. We are constructing a family of m 2-convex sets such that any subfamily has nonempty intersection yet there is no point common to all of them. Let Q_m be a regular polygon with m edges e_1, \dots, e_m (refer to Figure 17). Let P_i^* be the polygonal chain obtained from the boundary of Q_m by removing edge e_i and an infinitesimal portion of e_{i-1} and e_{i+1} . Finally, let us give some slight thickness to the chain so it becomes a polygon P_i . Notice that the polygons P_1, \dots, P_m are 2-convex, the intersection $\bigcap_{i=1}^m P_i$ is clearly empty, while the intersection of every proper subfamily F is nonempty because it contains the intervals e_j for all those $P_j \notin F$. \square

The preceding lemmas apply to k -convex sets in general, not only polygonal domains. (A *polygonal domain* is a set obtained by a finite number of union operations of simple polygons.) However, a significant difference appears in our next results, that are possibly

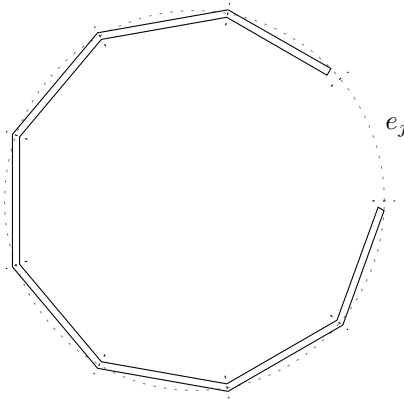


Figure 17: No Helly-type theorem for k -convex sets

the most natural to explore, because they involve transversal lines, which are precisely the main concept underlying the definition of k -convexity.

Let us recall [36] that given a family of sets Q_1, \dots, Q_m , a line ℓ is said to be a *transversal* of the family if ℓ has a nonempty intersection with each of the sets. When the sets are convex, the ordering in which they are traversed (disregarding the orientation of the line) is called a *geometric permutation*, a topic that has received significant attention [36]. In particular, it has been proved that m compact disjoint convex sets admit at most $2m - 2$ geometric permutations, which is tight [10].

Let us consider now transversals of compact 2-convex sets. Notice that every object will appear at least once, but may appear twice on the transversal, which we consider as combinatorially different cases of the associated *generalized geometric permutation*. What is the maximum number of generalized geometric permutations a family of m 2-convex sets may have?

Theorem 6. *The number of generalized geometric permutations of a set of m 2-convex objects may be exponential in m .*

Proof. We first show that a 2-convex objects can be arbitrarily complex. A *nose* of an object O is a zig-zag sequence of a reflex and a convex vertex of the boundary of O as depicted in Figure 18(a). Locally a line 'normal' to the nose intersects O in two connected components. The shaded area in Figure 18(a) indicates the region which is not intersected by a line tangent to one of the vertices of the nose. Thus we can iteratively construct further noses in this region without destroying the 2-convexity of O . Figure 18(b) shows an example where the principle shape of O is part of a disk. Observe that when the radius of the disk is large enough we can arrange an arbitrary number of flat noses such that O stays 2-convex.

Let R_i be an object which has the base shape of an axis-aligned rectangle, where the left side is actually part of a circle with sufficiently large radius and a center point far to the right of R_i . We place 2^{i-1} noses along this side, so that R_i stays 2-convex as described above. Next we arrange k objects R_1 to R_k from left to right, as depicted in Figure 18(c) for $k = 3$. We position the noses for each R_i in a regular way such that a rotating line

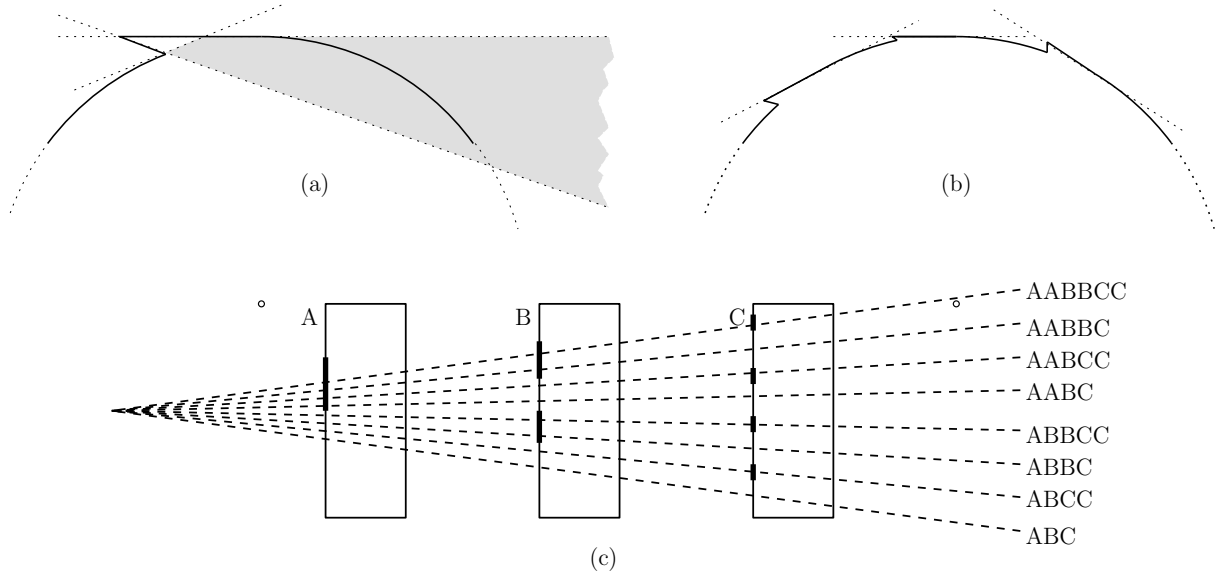


Figure 18: (a) and (b): 2-convex objects can have an unbounded number of noses. (c): The number of generalized geometric permutations for a set of 2-convex objects can be exponential.

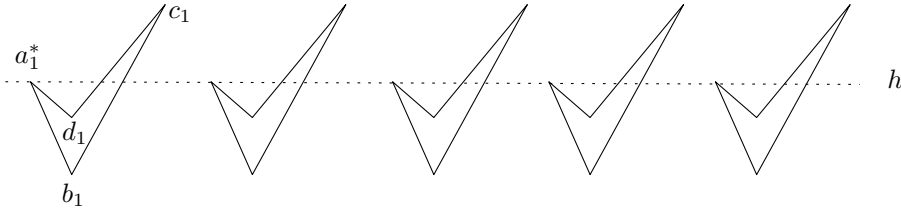


Figure 19: The number of generalized geometric permutations for a set of 2-convex polygons can be quadratic.

(see the dashed lines in Figure 18(c)) intersects the noses in the same manner as the digit “1” shows up in the sequence of all 2^k binary numbers of length k . Thus, we get 2^k different generalized geometric permutations for this setting, as each object appears twice if the nose is intersected, but only once otherwise. \square

Theorem 7. *The maximum number of generalized geometric permutations of a set of 2-convex polygonal domains with a total of n boundary edges is $\Theta(n^2)$.*

Proof. As the standard duality transform maps each edge to a double wedge, the induced arrangement of $2n$ lines in the dual plane yields a quadratic number of cells that bound from above the number of possible ways of stabbing the set of objects.

To see that the bound is asymptotically tight, we give a construction using n 2-convex polygons; in fact, the simplest possible ones, namely, nonconvex quadrilaterals. Let $Q_1^* = a_1^*b_1c_1d_1$ be the quadrilateral shown in Figure 19. Let h be a horizontal line through a_1^* and let $Q_i^* = a_i^*b_ic_id_i$ be translates of Q_1 in such a way that all of them

are pairwise disjoint and $a_1^*, a_2^*, \dots, a_n^*$ appear in this order on h . Finally, let us perturb infinitesimally a_i^* to a_i in such a way that

- a) points a_1, a_2, \dots, a_n are in general position,
- b) for all i, j, k , with $i \neq j$, the line $a_i a_j$ leaves above the point c_k and below the point d_k .

For $i = 1, \dots, n$, let Q_i be the quadrilateral with vertices $a_i b_i c_i d_i$. There are $\binom{n}{2}$ lines of the type $a_i a_j$; each of them leaves above and below a different set of points a_k , and is a transversal because it crosses all the segments $b_k c_k$ for every k . Now, if a_k is below the transversal, Q_k is intersected once, while if a_k is above the transversal, Q_k is intersected twice. Therefore, we have obtained $\binom{n}{2}$ generalized geometric permutations. \square

Observe that the two preceding theorems apply *mutatis mutandis* to k -convex sets, because 2-convex sets are also k -convex for $k \geq 3$.

5 Discussion and Open Problems

In this paper we have considered a new concept of generalized convexity. Moving from convexity to 2-convexity is seemingly a small change, as we are just accepting lines to intersect in at most two connected components instead of one. It is remarkable that this modest departure has strong consequences in the complexity of the new class of objects, as we have seen in this paper; obviously, even more when the degree of convexity is increased. Several open problems remain and many interesting questions can be raised. We list some of them below.

- a) Can the recognition of 2-convex polygons be carried out in linear time, improving on the $O(n \log n)$ algorithm we provide?
- b) Finding the smallest k such that a given polygon is k -convex is a 3SUM-hard problem. In particular, recognizing 4-convexity is already 3SUM-hard. We do not know whether the situation is the same for 3-convexity or whether a subquadratic time algorithm exists for this case.
- c) Is it possible to generalize Theorem 4? For example, is it true that every k -convex polygon with n vertices has a large subset of vertices that are the vertices of a $(k - 1)$ -convex polygon?
- d) Let us define the k -convex hull of a point set S as the smallest area polygon which is k -convex, has a subset $T \subset S$ as vertex set and every point in $S \setminus T$ is inside the polygon. Which is the complexity of computing this k -convex hull? Observe that for $k = 1$ this notion is the usual convex hull of a point set.
- e) Give combinatorial bounds and efficient algorithms for decomposing a polygon into k -convex subpolygons. This is a classical problem when convex subpolygons are considered [18] and also the decomposition into pseudotriangles, which are 2-convex polygons, has been studied [28]. However, the latter result might be improved by considering more general 2-convex polygons.

- f) A k -convex decomposition of a set S of n points in the plane is a decomposition of its convex hull into k -convex polygons such that every point in S is a vertex of some of the polygons. For $k = 1$, a triangulation suffices, though it has been proved that if we allow arbitrary convex sets the number can be reduced [24]. On the other hand, it has been shown that there exist always a decomposition into exactly $n - 2$ pseudotriangles, which are 2-convex polygons [28]. It is an intriguing open problem to decide whether this number can be reduced to sublinear if we allow arbitrary 2-convex polygons.

Finally, let us mention that in this paper we have focused on k -convex *polygons*. It is natural to define a similar concept for finite point sets, namely, being in k -convex *position*, where k is given by the smallest degree of convexity attained when *all* possible polygonizations of the point set are considered. This issue is considered in a companion paper [2].

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References

- [1] O. Aichholzer, R. Fabila-Monroy, D. Flores-Peñaloza, T. Hackl, C. Huemer, J. Urrutia, and B. Vogtenhuber. Modem Illumination of Monotone Polygons. In Proc. 25th European Workshop on Computational Geometry EuroCG '09, Brussels, Belgium, 2009, 167-170.
- [2] O. Aichholzer, F. Aurenhammer, T. Hackl, F. Hurtado, P.A. Ramos, J. Urrutia, P. Valtr, B. Vogtenhuber. *k-convex points sets*. Manuscript, 2010.
- [3] O. Aichholzer, C. Huemer, S. Kappes, B. Speckmann, C.D. Tóth. *Decompositions, partitions, and coverings with convex polygons and pseudo-triangles*. Graphs and Combinatorics 23 (2007), 481-507.
- [4] G. Aloupis, P. Bose, V. Dujmovic, C. Gray, S. Langerman, B. Speckmann. *Triangulating and guarding realistic polygons*. Proc. 20th Canadian Conference on Computational Geometry, 2008.
- [5] D. Artigas, M.C. Dourado, J.L. Szwarcfiter. *Convex partitions of graphs*. Electronic Notes in Discrete Mathematics 29 (2007), 147-151.
- [6] M. Breen, D.C. Kay. *General decomposition theorems for m -convex sets in the plane*. Israel Journal of Mathematics 24 (1976), 217-233.
- [7] M.R. Brown, R.E. Tarjan. *Design and Analysis of a Data Structure for Representing Sorted Lists*. SIAM Journal on Computing 9(3) (1980), 594-614.
- [8] B. Chazelle, H. Edelsbrunner, M. Grigni, L. Guibas, J. Hershberger, M. Sharir, J. Snoeyink. *Ray shooting in polygons using geodesic triangulations*. Algorithmica 12 (1994), 54-68.

- [9] B. Chazelle, J. Incerpi. *Triangulation and shape-complexity*. ACM Transactions on Graphics 3 (1984), 135-152.
- [10] H. Edelsbrunner, M. Sharir. *The maximum number of ways to stab n convex non-intersecting sets in the plane is $2n - 2$* Discrete Comput. Geom. 5 (1990), 35-42.
- [11] H. ElGindy, G.T. Toussaint. *On geodesic properties of polygons relevant to linear time triangulation*. The Visual Computer 5 (1989), 68-74.
- [12] P. Erdős, G. Szekeres. *A combinatorial problem in geometry*. Compositio Mathematica, 2 (1935), 463-470.
- [13] R. Fabila-Monroy, Andres Ruis Vargas, and J. Urrutia. *On Modern Illumination Problems*. XIII Encuentros de Geometria Computacional, Zaragoza, España, June 29 - July 1 2009, 9-19.
- [14] S.P. Fekete, M.E. Lübbecke, H. Meijer. *Minimizing the stabbing number of matchings, trees, and triangulations*. Proc. 15th Ann. ACM-SIAM Symp. on Discrete Algorithms, 2004, 430-439.
- [15] E. Fink, D. Wood. *Fundamentals of restricted-orientation convexity*. Information Sciences 92 (1996), 175-196.
- [16] A. Fournier, D.Y. Montuno. *Triangulating simple polygons and equivalent problems*. ACM Transactions on Graphics 3 (1984), 153-174.
- [17] A. Gajentaan and M. H. Overmars. *On a class of $O(n^2)$ problems in computational geometry*. CGTA: Computational Geometry: Theory and Applications, 5, (1995), 165-185.
- [18] M. Keil. *Polygon Decomposition* in Handbook of Computational Geometry (2000), 491-518.
- [19] J. King. *A Survey of 3SUM-Hard Problems*. <http://www.cs.mcgill.ca/~jking/>
- [20] G. Kun, G. Lippner. *Large empty convex polygons in k -convex sets*. Periodica Mathematica Hungarica 46 (2003), 81-88.
- [21] D.T. Lee, F.P. Preparata. *An optimal algorithm for finding the kernel of a polygon*. Journal of the ACM 26 (1979), 415-421.
- [22] A. Maheshwari, J.-R. Sack, H. Djidjev. *Link distance problems*. In: Handbook of Computational Geometry, J.-R. Sack and J. Urrutia (eds.), Elsevier, 2000, 519-558.
- [23] J. Matoušek, P. Plecháč. *On functional separately convex hulls*. Discrete & Computational Geometry 19 (1998), 105-130.
- [24] V. Neumann-Lara, E. Rivera-Campo, J. Urrutia. *A note on convex decompositions of point sets in the plane*. Graphs and Combinatorics 20(2) (2004), 223-231.
- [25] J. O'Rourke. *Art Gallery Theorems and Algorithms*. Oxford University Press, 1987.

- [26] T. Pennanen. *Graph-convex mappings and K -convex functions*. Journal of Convex Analysis 6 (1999), 235-266.
- [27] G.J.E. Rawlins, D. Wood. *Ortho-convexity and its generalizations*. In: Computational Morphology, G.T. Toussaint (ed.), North-Holland Publishing Co., Amsterdam, 1988, 137-152.
- [28] G. Rote, F. Santos, I. Streinu. *Pseudo-triangulations—a survey*. Contemporary Mathematics 453 (2008), 343-410.
- [29] S. Schuierer, D. Wood. *Visibility in semi-convex spaces*. Journal of Geometry 60 (1997), 160-187.
- [30] D.D. Sleator, R.E. Tarjan. *Self-adjusting binary search trees*. Journal of the ACM 32(3) (1985), 652-686.
- [31] J. Urrutia. *Art Gallery and Illumination Problems*. Handbook on Computational Geometry, Elsevier Science Publishers, J.R. Sack and J. Urrutia eds. pp. 973-1026, 2000.
- [32] F.A. Valentine. *A three-point convexity property*. Pacific Journal of Mathematics 7 (1957), 1227-1235.
- [33] P. Valtr. *A sufficient condition for the existence of large empty convex polygons*. Discrete Comput. Geom., 28 (2002), 671-682.
- [34] G. Vegter. *Kink-free deformations of polygons*. Proc. 5th Ann. ACM Symp. Computational Geometry, 1989, 61-68.
- [35] E. Welzl. *On spanning trees with low crossing numbers*. In: Data Structures and Efficient Algorithms, B. Monien and T. Ottmann (eds.), Springer LNCS 594, 1992, 233-249.
- [36] R. Wenger, *Progress in geometric transversal theory*. Contemporary Mathematics, B. Chazelle and J.E. Goodman, Eds., American Math Society (1999), 375-393.

